From asymptotics to spectral measures: determinate versus indeterminate moment problems

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Abstract

In the field of orthogonal polynomials theory, the classical Markov theorem shows that for determinate moment problems the spectral measure is under control of the polynomials asymptotics.

The situation is completely different for indeterminate moment problems, in which case the interesting spectral measures are to be constructed using Nevanlinna theory. Nevertheless it is interesting to observe that some spectral measures can still be obtained from weaker forms of Markov theorem.

The exposition will be illustrated by orthogonal polynomials related to elliptic functions: in the determinate case by examples due to Stieltjes and some of their generalizations and in the indeterminate case by more recent examples.

1 Background material

Let us consider the three terms recurrence [1]

$$xP_n = b_{n-1}\,P_{n-1} + a_n P_n + b_n\,P_{n+1}, \qquad n \geq 1. \tag{1}$$

We will denote by P_n and Q_n two linearly independent solutions of this recurrence with initial conditions

$$P_0(x) = 1, \qquad P_1(x) = \frac{x - a_0}{b_0}, \qquad\qquad Q_0(x) = 0, \qquad Q_1(x) = \frac{1}{b_0}. \tag{2}$$

The corresponding Jacobi matrix is

$$\begin{pmatrix} a_0 & b_0 & 0 & 0 & 0 & \cdots \\ b_0 & a_1 & b_1 & 0 & 0 & \cdots \\ 0 & b_1 & a_2 & b_2 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \tag{3}$$

If the $b_n > 0$ and $a_n \in \mathbb{R}$ the P_n (resp. the Q_n) will be orthogonal with respect to a positive probabilistic measure ψ (resp $\psi^{(1)}$)

$$\int P_m(x)\,P_n(x)\,d\psi(x) = \delta_{mn}, \qquad\qquad \int Q_m(x)\,Q_n(x)\,d\psi^{(1)}(x) = \delta_{mn}, \tag{4}$$

with the moments

$$s_n = \int x^n\,d\psi(x), \qquad n \geq 0, \qquad s_0 = 1 \tag{5}$$

If $\mathrm{supp}\,\psi \subset [0, +\infty[$ we have a Stieltjes moment problem while if $\mathrm{supp}\,\psi \subset]-\infty, +\infty[$ we have a Hamburger moment problem. These moment problems may be determinate (det S or det H) if the measure is unique or indeterminate (indet S or indet H) if it is not unique.

For further use, we will introduce new polynomials $F_n(x)$ by

$$P_n(x) = \frac{(-1)^n}{\sqrt{\pi_n}}\,F_n(x), \qquad n \geq 0, \qquad \begin{array}{ll} a_n = \lambda_n + \mu_n, & b_n = \sqrt{\lambda_n \mu_{n+1}}, \quad n \geq 0, \\ \pi_0 = 1, & \pi_n = \dfrac{\lambda_0 \lambda_1 \ldots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n}, \quad n \geq 1, \end{array} \tag{6}$$

From (1) we deduce

$$\begin{aligned} & -xF_n = \mu_{n+1} F_{n+1} + (\lambda_n + \mu_n) F_n + \lambda_{n-1} F_{n-1}, \\ & F_{-1}(x) = 0, \quad F_0(x) = 1, \end{aligned} \tag{7}$$

[1] We stick, as far as possible, to Akhiezer's notations in [1].

Similarly, defining

$$Q_n(x) = \frac{(-1)^{n-1}}{\mu_1\sqrt{\pi_n}}\, F^{(1)}_{n-1}(x), \tag{8}$$

one can check from (1) that the $F^{(1)}_n(x)$ are a solution of the recurrence (7) with the substitution $(\lambda_n,\, \mu_n) \;\rightarrow\; (\lambda_{n+1}, \mu_{n+1})$, i. e. the associated polynomials of order one.

Notice the useful relations and notations, valid for $\mu_0 = 0$, easily derived by induction

$$P_n(0) = (-1)^n\sqrt{\pi_n}, \qquad Q_n(0) = (-1)^n\frac{\sqrt{\pi_n}}{\alpha_n}, \qquad \frac{1}{\alpha_n} = -\sum_{k=1}^{n}\frac{1}{\mu_k\pi_k}. \tag{9}$$

The determinate case

2 Markov theorem

In the determinate case (det H hence det S), given (a_n, b_n) the basic tool to compute the spectral measure is Markov theorem. In the classical textbooks [20, §3.5],[6, p. 89] it is proved under the restrictive assumption that the measure support is bounded (which implies that the moment problem is determinate). More recently it was proved under the sole hypothesis of determinacy of the moment problem [3],[24]. It can be stated as :

Proposition 1 *For a determinate moment problem the Stieltjes transform of the (unique) orthogonality measure is given by*

$$\lim_{n\to\infty}\frac{Q_n(x)}{P_n(x)} = -\lim_{n\to\infty}\frac{F^{(1)}_{n-1}(x)}{\mu_1\, F_n(x)} = \int\frac{d\psi(t)}{x-t}, \qquad x\in V=\mathbb{C}\backslash\mathbb{R}, \tag{10}$$

where the convergence is uniform in compact subsets of V.

Let us mention the connection with finite continued fractions. One has

$$\frac{Q_n(x)}{P_n(x)} = 1/x - a_0 - b_0^2/x - a_1 - b_1^2/x - \cdots - b_{n-2}^2/x - a_{n-1}, \tag{11}$$

which can be written, using the Christoffel numbers $\lambda_{k,n}$ according to

$$\frac{Q_n(x)}{P_n(x)} = \sum_{k=1}^{n}\frac{\lambda_{k,n}}{x-x_{k,n}} = \int\frac{d\psi_n(t)}{x-t}. \tag{12}$$

As shown in [3], when the moment problem is determinate, the measure ψ_n converges weakly to ψ. The limiting continued fraction does give the Stieltjes transform of the spectral measure

$$\int\frac{d\psi(t)}{x-t} = 1/x - a_0 - b_0^2/x - a_1 - b_1^2/x - \cdots, \qquad x\in\mathbb{C}\backslash\mathbb{R}. \tag{13}$$

The continued fraction encodes not only the coefficients appearing in the recurrence relation of the polynomials, but also the moments in the asymptotic series

$$\int \frac{d\psi(t)}{x+t} \asymp \sum_{n\geq 0}(-1)^n \frac{s_n}{x^{n+1}}, \tag{14}$$

valid uniformly in $\delta \leq \arg x \leq \pi - \delta$ provided that $0 < \delta < \pi/2$, as shown in [1, p. 95].

Since we want to discuss some work of Stieltjes, let us mention that he often substitutes

$$x \longrightarrow -x^2, \qquad a_n = \lambda_n + \mu_n, \qquad b_n^2 = \lambda_n \, \mu_{n+1},$$

and writes

$$\int \frac{d\psi(t)}{x^2+t} = 1/x^2 + \lambda_0 + \mu_0 - \lambda_0\mu_1/x^2 + \lambda_1 + \mu_1 - \lambda_1\mu_2/x^2 + \cdots, \qquad x \in \mathbb{C}\backslash\mathbb{R}. \tag{15}$$

Let us make the further assumption that $\mu_0 = 0$. Considering

$$\int \frac{x}{x^2+t} d\psi(t) = x(1/x^2 + \lambda_0 - \lambda_0\mu_1/x^2 + \lambda_1 + \mu_1 - \lambda_1\mu_2/x^2 + \cdots),$$

and upon iteration of the identity [19, p. 404]

$$x^2 + \lambda_0/1 + \mu_1/D = x^2 + \lambda_0 - \lambda_0\mu_1/\mu_1 + D,$$

we get first

$$\int \frac{x}{x^2+t} d\psi(t) = 1/x^2 + \lambda_0/1 + \mu_1/x^2 + \lambda_1/1 + \mu_2/x^2 + \cdots,$$

easily transformed into

$$\int \frac{x}{x^2+t} d\psi(t) = 1/x + \lambda_0/x + \mu_1/x + \lambda_1/x + \mu_2/x + \cdots \tag{16}$$

So, under the assumption that $\mu_0 = 0$, we have transformed the initial J-continued fraction (13) into an S-continued fraction (16), following the derivation due to Stieltjes in [19].

3 Stieltjes continued fractions with elliptic functions

Stieltjes gave four continued fractions involving the Jacobi elliptic functions usually denoted as $\mathrm{sn}\,(u, k^2)$, $\mathrm{cn}\,(u, k^2)$ and $\mathrm{dn}\,(u, k^2)$, with parameter $0 < k^2 < 1$. Let us record two of them

$$\begin{aligned} \int_0^\infty \mathrm{dn}\, u\, e^{-xu}\, du &= 1/x + k^2/x + 2^2/x + 3^2k^2/x + 4^2/x + \cdots, \\ \int_0^\infty \mathrm{cn}\, u\, e^{-xu}\, du &= 1/x + 1^2/x + 2^2k^2/x + 3^2/x + 4^2k^2/x + \cdots \end{aligned} \tag{17}$$

for $Re\, x > 0$. These relations are also quoted in Wall's book [25, §94].

On these relations we recognize S-continued fractions, the first one corresponding to the polynomials with recurrence coefficients $\lambda_n = k^2(2n+1)^2 \quad \mu_n = 4n^2$, and the second one to $\lambda_n = (2n+1)^2 \quad \mu_n = 4k^2n^2$.

Notice that using the transformation theory of elliptic functions, namely the relation $\mathrm{dn}\,(u;k) = \mathrm{cn}\,(ku;1/k)$, one can deduce, by elementary algebra, the second continued fraction from the first one.

The first proof of (17), published by Stieltjes in 1889 in [18], used intensively the addition relations for the elliptic Jacobi functions and was quite lengthy (it may be found in Wall's book). But in 1891, in a letter to Hermite (published only in 1905 [2, p. 208]), he found an elegant shorter proof which we shall report [2]

The starting point is to define

$$C_n = \int_0^\infty \mathrm{cn}\,u\,(\mathrm{sn}\,u)^n\,e^{-xu}\,du, \qquad D_n = \int_0^\infty \mathrm{dn}\,u\,(\mathrm{sn}\,u)^n\,e^{-xu}\,du, \qquad n \in \mathbb{N}. \tag{18}$$

For $\mathrm{Re}\,x > 0$ an integration by parts gives

$$\begin{aligned} &xC_0 = 1 - D_1, \qquad && xC_n = nD_{n-1} - (n+1)D_{n+1}, \qquad && n \geq 1, \\ &xD_0 = 1 - k^2 C_1, \qquad && xD_n = nC_{n-1} - k^2(n+1)C_{n+1}, \qquad && n \geq 1. \end{aligned} \tag{19}$$

So if we define

$$p_0 = C_0, \qquad p_n = \frac{C_n}{nD_{n-1}}, \qquad q_0 = D_0, \qquad q_n = \frac{D_n}{nC_{n-1}}, \quad n \geq 1, \tag{20}$$

we get the non-linear recurrences

$$p_n = \frac{1}{x + (n+1)^2\,q_{n+1}}, \qquad q_n = \frac{1}{x + k^2(n+1)^2\,p_{n+1}}, \qquad n \geq 0. \tag{21}$$

Iterating these relations starting from p_0 and q_0 gives relations (17).

The continued fractions given by Stieltjes are quite impressive, since from them we can get easily the moments and the orthogonality measure, as we will explain now.

Let us start from the Taylor series

$$\begin{cases} \mathrm{dn}\,u = \displaystyle\sum_{n\geq 0} (-1)^n \frac{s_n}{(2n)!}\,u^{2n}, \\ s_0 = 1, \qquad s_1 = k^2, \qquad s_2 = k^2(4+k^2), \qquad s_3 = k^2(16 + 44k^2 + k^4), \qquad \cdots \end{cases} \tag{22}$$

which, inserted in (17), induces the asymptotic series

$$\int \frac{x}{x^2+t}\,d\psi(t) = \int_0^\infty \mathrm{dn}\,u\,e^{-xu}\,du \asymp \sum_{n\geq 0} (-1)^n \frac{s_n}{x^{2n+1}}, \tag{23}$$

from which we conclude that the coefficients s_n are indeed the moments of ψ. Their asymptotics follows easily from the generating function (22) and Darboux theorem:

$$s_n \underset{n\to\infty}{\sim} 2\frac{(2n)!}{(K')^{2n+1}}, \tag{24}$$

showing explicitly that the series (23) is indeed asymptotic.

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[2]Exactly the same proofs appear in [15], without any reference to Stieltjes, but some years later, in 1907.

Let us start from the Fourier series

$$\operatorname{dn} u = \psi_0 + \sum_{n\geq 1} \psi_n \cos\left(n\frac{\pi u}{K}\right), \tag{25}$$

with the coefficients

$$\psi_0 = \frac{\pi}{2K}, \qquad \psi_n = \frac{2\pi}{K}\,\frac{q^n}{1+q^{2n}}, \quad n \geq 1, \qquad q = e^{-\pi K'/K}. \tag{26}$$

Inserting this relation into the first continued fraction (17) gives

$$\int \frac{x}{x^2+t}\, d\psi(t) = \frac{\psi_0}{x} + \sum_{n\geq 1} \psi_n \, \frac{x}{x^2+(n\pi/K)^2}, \tag{27}$$

showing that the spectral measure is discrete

$$\psi = \sum_{n\geq 0} \psi_n \, \epsilon_{(n\pi/K)^2}, \tag{28}$$

where ϵ_s is the discrete measure with unit jump. Similar results can be obtained for the first continued fraction in (17).

These deep and elegant results of Stieltjes are quite frustrating since they apparently don't bear any relation with asymptotics. So how should we proceed to derive Stieltjes results using Markov theorem?

4 Stieltjes continued fractions from Markov theorem

Let us consider the continued fraction with $\lambda_n = k^2(2n+1)^2$ and $\mu_n = 4n^2$. We need the asymptotics of the polynomials F_n and of their associates of order one $F_n^{(1)}$. So we need *two* generating functions. Carlitz [5] has obtained a first one

$$F(x;w) \equiv \sum_{n\geq 0} \frac{n!}{(1/2)_n} F_n(x)\, w^n = \frac{\cos(\sqrt{x}\theta(w))}{\sqrt{1-k^2 w}}, \qquad \theta(w) = \int_0^w \frac{du}{2\sqrt{u(1-u)(1-k^2u)}}. \tag{29}$$

Notice, en passant, that $G(x;w) = \sqrt{1-k^2w}\, F(x;w)$ is a solution of Heun's differential equation [16]

$$\frac{d^2G}{dw^2} + \left(\frac{1/2}{w} - \frac{1/2}{1-w} - \frac{k^2/2}{1-k^2w}\right)\frac{dG}{dw} + \frac{x}{4}\,G = 0. \tag{30}$$

Using theorem (8.4) in [20] (see [21] for the details) one deduces the asymptotics

$$F_n(x) \sim -\frac{1}{2k'^2\, n}\,\frac{\pi_n}{(k^2)^n}\,\sqrt{x}\sin(\sqrt{x}K), \qquad x \in \mathbb{C}\backslash\mathbb{R}. \tag{31}$$

The generating function needed for the associated polynomials $F_n^{(1)}$ was given in [21] (set $c=1$ and $\mu=0$ in the relation (2.15) of this reference):

$$\sum_{n\geq 0} \frac{(2)_n}{(3/2)_n} w^{n+1}\, \frac{F_n^{(1)}(x)}{\mu_1} = \frac{N(w)}{2\sqrt{1-k^2w}}, \tag{32}$$

with

$$N(w)=\int_0^{\theta(w)}\frac{\sin(\sqrt{x}(\theta(w)-u))}{\sqrt{x}}\,\mathrm{dn}\,u\,du. \tag{33}$$

Darboux theorem gives

$$\frac{F_n^{(1)}(x)}{\mu_1}\sim-\frac{1}{2k'^2\,n}\,\frac{\pi_n}{(k^2)^n}\int_0^K\cos(\sqrt{x}(K-u))\,\mathrm{dn}\,u\,du,\qquad x\in\mathbb{C}\backslash\mathbb{R}. \tag{34}$$

We can now use Markov theorem to obtain

$$\int\frac{d\psi(t)}{x-t}=\frac{\int_0^K\,\mathrm{dn}\,u\cos(\sqrt{x}(K-u))\,du}{\sqrt{x}\sin(\sqrt{x}K)},\qquad x\in\mathbb{C}\backslash\mathbb{R}.$$

Let us reduce this result to its Stieltjes form. We first substitute $x\Rightarrow -x^2$ which gives

$$\int\frac{x}{x^2+t}d\psi(t)=\frac{1}{\sinh(xK)}\int_0^K\,\mathrm{dn}\,u\cosh(x(K-u))\,du.$$

The change of variables $v=2K-u$ allows to show

$$\int_0^K e^{-x(K-u)}\,\mathrm{dn}\,u\,du=\int_K^{2K}e^{x(K-v)}\,\mathrm{dn}\,v\,dv,$$

and this implies

$$\int_0^K\,\mathrm{dn}\,u\cosh(x(K-u))\,du=e^{xK}\int_0^{2K}e^{-xu}\,\mathrm{dn}\,u\,du.$$

It follows for the Stieltjes transform that

$$\int\frac{x}{x^2+t}d\psi(t)=\frac{1}{1-e^{-2xK}}\int_0^{2K}e^{-xu}\,\mathrm{dn}\,u\,du=\int_0^{\infty}\,\mathrm{dn}\,u\,e^{-xu}\,du,\qquad \mathrm{Re}\,x>0.$$

The last equality follows from the $2K$-periodicity of $\mathrm{dn}\,u$. So, quite satisfactorily, Markov theorem reproduces Stieltjes results, certainly not so elegantly, but with the possibility of some generalizations which would be quite difficult remaining in Stieltjes approach.

5 Generalization of Stieltjes results

Since Stieltjes results in the nineteenth century, only a few generalizations could be obtained. The first one is due to the Chudnowski [8], who changed the elliptic function $f(u)=\mathrm{dn}\,u$ into solutions of Lamé's equation

$$\frac{d^2f}{du^2}+xf=n(n+1)k^2\,\mathrm{sn}^2\,u\,f,\qquad n\in\mathbb{N},$$

but no explicit results were given on the spectral measure and, since n is an integer, there is no limiting process which can lead back to Stieltjes continued fractions (17).

Another generalization, involving a continuous parameter $c>0$, was obtained in [21]. Working out an appropriate generating function and the polynomials asymptotics, Markov theorem [3] yields :

[3] Use relations given page 756 in the previous reference, and algebraic steps as in section 4.

Proposition 2 *For the orthogonal polynomials with recurrence coefficients*

$$\lambda_n = k^2(2n+2c+1)^2, \qquad \mu_n = 4(n+c)^2(1-\delta_{n0}), \qquad n \geq 0, \tag{35}$$

the Stieltjes transform of the orthogonality measure is given, for $x \in \mathbb{C}\backslash\mathbb{R}$ *and* $c > 0$, *by*

$$\int \frac{x}{x^2+t} d\psi(t) = \frac{N(c;x)}{D(c;x)} = 1/x + \lambda_0/x + \mu_1/x + \lambda_1/x + \mu_2/x + \cdots, \tag{36}$$

with

$$N(c;x) = \int_0^{2K} \mathrm{dn}\, u\, \frac{(\mathrm{sn}\, u)^{2c}}{(2c)!}\, e^{-xu}\, du, \qquad D(c;x) = \int_0^{2K} \mathrm{cn}\, u\, \frac{(\mathrm{sn}\, u)^{2c-1}}{(2c-1)!}\, e^{-xu}\, du, \tag{37}$$

using the notation $(\alpha)! = \Gamma(\alpha+1)$.

Remarks:

1. The limit $c \to 0$ is tricky for D. One has to use

$$\lim_{c\to 0} D(c;x) = \lim_{c\to 0} 2e^{-xK} \int_0^K \sinh(x(K-u))\, \mathrm{cn}\, u\, \frac{(\mathrm{sn}\, u)^{2c-1}}{(2c-1)!}\, du = 2e^{-xK} \sinh(xK),$$

and in that way Stieltjes result is recovered, but we see that for a generic value of c it is no longer possible to transform this ratio of integrals into a single integral.

2. Their spectral properties are now under investigation [17]: it can be shown that the spectrum is discrete and that its asymptotic behaviour is independent of the parameter c.

The indeterminate case

6 The Nevanlinna parametrization

According to the growth of the coefficients (λ_n, μ_n), with $\mu_0 = 0$, we may have three different possibilities [1]:

1. indet S iff $\sum_{n=1}^{\infty} (\pi_n + 1/\mu_n\pi_n) < \infty$.
2. indet H (which implies indet S) iff $\sum_{n=1}^{\infty} \pi_n (\sum_{k=1}^{n} 1/\mu_k\pi_k)^2 < \infty$.
3. det S and indet H iff $\sum_{n=1}^{\infty} 1/\mu_n\pi_n = \infty$ and $\sum_{n=1}^{\infty} \pi_n (\sum_{k=1}^{n} 1/\mu_k\pi_k)^2 < \infty$.

For an indeterminate moment problem (see a detailed account in [4]), one first defines the series

$$\begin{aligned} A_n(x) &= x\sum_{k=0}^{n-1} Q_k(0)\, Q_k(x), & C_n(x) &= 1 + x\sum_{k=0}^{n-1} P_k(0)\, Q_k(x), \\ B_n(x) &= -1 + x\sum_{k=0}^{n-1} Q_k(0)\, P_k(x), & D_n(x) &= x\sum_{k=0}^{n-1} P_k(0)\, P_k(x), \end{aligned} \tag{38}$$

constrained by

$$A_n(x)D_n(x) - B_n(x)C_n(x) = 1.$$

In the indet H case, these series, for $n \to \infty$, converge absolutely and uniformly [1] on compact subsets of $\mathbb{C}$) to entire functions $A(x), \cdots, D(x)$.

The Nevanlinna matrix $\mathcal{N}$ is then

$$\mathcal{N}(x) = \begin{pmatrix} A(x) & C(x) \\ B(x) & D(x) \end{pmatrix}, \qquad A(x)D(x) - B(x)C(x) = 1, \qquad \forall x \in \mathbb{C}. \tag{39}$$

It gives the Stieltjes transform of all the Nevanlinna-extremal (or N-extremal) measures

$$\int \frac{d\psi_\lambda(t)}{x-t} = \frac{A(x)\lambda - C(x)}{B(x)\lambda - D(x)}, \qquad \lambda \in \mathbb{R} \cup \{\infty\} \sim S^1. \tag{40}$$

For these measures and only for these measures are the polynomials P_n dense in $L^2(\mathbb{R}, d\psi_\lambda)$.

Let us observe that the Stieltjes transform being meromorphic, the N-extremal measures are all discrete with

$$\psi_\lambda = \sum_{s \in Z_\lambda} \psi_\lambda(s)\, \epsilon_s, \qquad \psi_\lambda(s) = \frac{1}{B'(s)D(s) - B(s)D'(s)} \tag{41}$$

where Z_λ is the zero set of the entire function $B\lambda - D$ (or B for $\lambda = \infty$).

The series

$$\frac{1}{\alpha_n} = \frac{Q_n(0)}{P_n(0)} = -\sum_{k=1}^{n} \frac{1}{\mu_k \pi_k}, \qquad \frac{1}{\alpha} = \lim_{n\to\infty} \frac{1}{\alpha_n} = -\sum_{k=1}^{\infty} \frac{1}{\mu_k \pi_k}, \tag{42}$$

is quite important since, as shown in [7], [4] the positively supported measures are given by $\lambda \in [\alpha, 0]$. As we will see the border measures ψ_0 and ψ_α play a prominent role. In terms of the self-adjoint extensions of the Jacobi matrix ψ_0 corresponds to Krein's extension [13] and ψ_α corresponds to Friedrichs extension [14].

Polynomials for which the Nevanlinna matrix and N-extremal measures are known, more or less explicitly, are not very numerous: they correspond to strong increase of the (λ_n, μ_n) for large n. This increase may be exponential, as for the q^{-1}-Hermite [11], and in this case all the N-extremal measures are known explicitly! Many other references to related to q-polynomials are given in [4].

When the (λ_n, μ_n) are some particular quartic polynomial [4] the Nevanlinna matrix and the border N-extremal measures are explicitly known. More recently the Nevanlinna matrices for some cubic cases have been obtained [9] but only the asymptotics of the N-extremal spectra could be obtained. An example of the "exotic" case det S and indet H is available for the Al-Salam-Carlitz polynomials and is discussed in [4].

Let us now turn to the determination of the Nevanlinna matrix from generating functions.

7 Dual polynomials versus Nevanlinna matrix

Using the relations given in (6), (8) and (9) the Nevanlinna matrix can be written as

$$\begin{cases} A(x) = -\dfrac{x}{\mu_1}\displaystyle\sum_{n=1}^{\infty}\frac{F_{n-1}^{(1)}(x)}{\alpha_n}, & B(x) = -1 + x\displaystyle\sum_{n=1}^{\infty}\frac{F_n(x)}{\alpha_n}, \\ C(x) = 1 - \dfrac{x}{\mu_1}\displaystyle\sum_{n=0}^{\infty} F_n^{(1)}(x), & D(x) = x\displaystyle\sum_{n=0}^{\infty} F_n(x). \end{cases} \tag{43}$$

If we know the generating function $G(x;w) = \sum_{n\geq 0} F_n(x)w^n$, from Abel's lemma we deduce

$$D(x) = x \lim_{w\to 1-} G(x;w),$$

and similarly for the function C related to the polynomials $F_n^{(1)}$.

The computation of A and B, as shown in [22], is related to the dual polynomials $\widetilde{F}_n$ defined in [12] by the recurrence

$$\begin{aligned} &-x\widetilde{F}_n = \widetilde{\mu}_{n+1}\widetilde{F}_{n+1} + (\widetilde{\lambda}_n + \widetilde{\mu}_n)\widetilde{F}_n + \widetilde{\lambda}_{n-1}\widetilde{F}_{n-1}, \\ &\widetilde{F}_{-1}(x) = 0, \quad \widetilde{F}_0(x) = 1, \end{aligned} \tag{44}$$

with the coefficients [12]

$$\widetilde{\lambda}_n = \mu_{n+1}, \quad n \geq 0, \qquad \widetilde{\mu}_n = \lambda_n, \quad n \geq 0, \qquad \widetilde{\pi}_n = \frac{\lambda_0}{\mu_{n+1}\pi_{n+1}}. \tag{45}$$

Notice that for the initial coefficients $(\lambda_n,\, \mu_n)$ we have $\mu_0 = 0$, but for the dual coefficients $\widetilde{\mu}_0 = \lambda_0 > 0$ from positivity.

Let us prove first:

Proposition 3 *Let us consider an indet S moment problem, with coefficients* $(\lambda_n,\, \mu_n)$ *such that* $\mu_0 = 0$. *Let the* $\widetilde{F}_n$ *be the dual polynomials as defined previously. Then one has*

$$B(x) - \frac{D(x)}{\alpha} = -1 + \frac{x}{\widetilde{\mu}_0}\sum_{n\geq 0}\widetilde{F}_n(x). \tag{46}$$

Proof:
Let us start from the double series for B given in (43). Since the moment problem is indet S, the series $-\dfrac{1}{\alpha_n}$ is absolutely convergent and the same is true for the series $\sum_n F_n(x)$ for x in any compact subset of $\mathbb{C}$. We can interchange the order of the summations to get

$$B(x) = -1 - x\sum_{k\geq 1}\frac{1}{\mu_k\pi_k}\sum_{n\geq k} F_n(x) = -1 - x\sum_{k\geq 1}\frac{1}{\mu_k\pi_k}\left(\sum_{n\geq 0} F_n(x) - \sum_{n=0}^{k-1} F_n(x)\right). \tag{47}$$

The first piece is related to the function D and the second one is simplified using the relation, proved by induction:

$$\sum_{n=0}^{k-1} F_n(x) = \frac{1}{\widetilde{\pi}_{k-1}}\,\widetilde{F}_{k-1}(x) = \frac{\mu_k\pi_k}{\widetilde{\mu}_0}\,\widetilde{F}_{k-1}(x), \tag{48}$$

and this concludes the proof. $\square$

Let us define the zero-related dual polynomials $\widehat{F}_n$ as those polynomials with recurrence coefficients

$$\widehat{\lambda}_n = \widetilde{\lambda}_n = \mu_{n+1}, \quad n \geq 0, \qquad \widehat{\mu}_n = \widetilde{\mu}_n(1-\delta_{n0}) = \lambda_n(1-\delta_{n0}). \tag{49}$$

These new polynomials can be expressed in terms of the $\widetilde{F}_n$ and their associates of order one by

$$\widehat{F}_n(x) = \widetilde{F}_n(x) - \frac{\widetilde{\mu}_0}{\widetilde{\mu}_1}\widetilde{F}^{(1)}_{n-1}(x), \qquad n \geq 0.$$

We are now in position to prove:

Proposition 4 *Let us consider an indet S moment problem, with coefficients* $(\lambda_n,\, \mu_n)$ *such that* $\mu_0 = 0$. *Let the* $\widehat{F}_n$ *be the zero-related dual polynomials as defined above. Then one has*

$$A(x) - \frac{C(x)}{\alpha} = \frac{1}{\widetilde{\mu}_0}\sum_{n\geq 0}\widehat{F}_n(x). \tag{50}$$

Proof:
Let us start from the double series for A given in (43). By the same arguments as in the previous proposition, we can interchange the order of the summations to get

$$A(x) = \frac{x}{\mu_1}\sum_{k\geq 1}\frac{1}{\mu_k\pi_k}\left(\sum_{n\geq 1}F^{(1)}_{n-1}(x) - \sum_{n=1}^{k-1}F^{(1)}_{n-1}(x)\right). \tag{51}$$

The first piece is related to the function C and the second one is simplified using the relation, proved by induction:

$$-\frac{x}{\mu_1}\sum_{n=1}^{k-1}F^{(1)}_{n-1}(x) = -1 + \frac{1}{\widetilde{\pi}_{k-1}}\widehat{F}_{k-1}(x), \tag{52}$$

and, taking into account $\mu_k\pi_k\widetilde{\pi}_{k-1} = \widetilde{\mu}_0$, this concludes the proof. $\square$

To conclude this section, it seems interesting to modify slightly the Nevanlinna matrix $\mathcal{N}$ to the form

$$\widetilde{\mathcal{N}}(x) = \begin{pmatrix} \widetilde{A}(x) & \widetilde{C}(x) \\ \widetilde{B}(x) & \widetilde{D}(x) \end{pmatrix}, \qquad \begin{array}{ll} \widetilde{A} = A - \dfrac{C}{\alpha}, & \widetilde{C} = C, \\ \widetilde{B} = B - \dfrac{D}{\alpha}, & \widetilde{D} = D, \end{array} \qquad \det\widetilde{\mathcal{N}} = 1. \tag{53}$$

Then the Stieltjes transform, defining $\mu = \alpha\lambda/(\lambda-\alpha)$, becomes

$$\int\frac{d\psi_\mu(t)}{x-t} = \frac{\widetilde{A}(x)\mu - \widetilde{C}(x)}{\widetilde{B}(x)\mu - \widetilde{D}(x)}, \qquad \mu \in \mathbb{R}\cup\{\infty\} \sim S^1. \tag{54}$$

The positively supported measures correspond now to $\mu \in \mathbb{R}^+\cup\{\infty\}$, and the border measures $(\psi_\alpha,\, \psi_0)$ become $(\psi_\infty,\, \psi_0)$.

8 Markov-like theorems

Despite Nevanlinna theory, which describes all the measures, the question of what survives from Markov theorem remains interesting . As we will see, the two border measures ψ_α and ψ_0 are still given by Markov-like theorems. Indeed one has first:

Proposition 5 *For an indeterminate Stieltjes moment problem we have*

$$\lim_{n\to\infty}\frac{Q_n(x)}{P_n(x)} = -\lim_{n\to\infty}\frac{1}{\mu_1}\frac{F^{(1)}_{n-1}(x)}{F_n(x)} = \int\frac{d\psi_\alpha(t)}{x-t}, \qquad x\in V=\mathbb{C}\backslash\mathbb{R} \tag{55}$$

where the convergence is uniform for x in any compact subset of V.

Proof:
The proof given in [3] follows easily from two relations proved in [1, p. 14], which may be written

$$Q_n(x) = Q_n(0)C_n(x) - P_n(0)A_n(x), \qquad P_n(x) = Q_n(0)D_n(x) - P_n(0)B_n(x). \tag{56}$$

We can replace $P_n(0)/Q_n(0)$ by α_n (see relation (9)) so that

$$\frac{Q_n(x)}{P_n(x)} = \frac{A_n(x)\,\alpha_n - C_n(x)}{B_n(x)\,\alpha_n - D_n(x)}. \tag{57}$$

For $n\to\infty$, since we are indet S, we have $\lim\limits_{n\to\infty}\alpha_n=\alpha$ and the series $A_n(x),\cdots,D_n(x)$ converge uniformly in compact subsets of V to the entire functions $A(x),\cdots,D(x)$. It follows that

$$\lim_{n\to\infty}\frac{Q_n(x)}{P_n(x)} = \frac{A(x)\,\alpha - C(x)}{B(x)\,\alpha - D(x)}. \tag{58}$$

The theorem follows from (40). $\Box$

Remark: If the moment problem is det S but indet H, then $\psi_\alpha=\psi_0$, is the *unique* measure supported by $[0,+\infty[$ (the previous theorem does still work in this case), while there are plenty of different measures supported by $\mathbb{R}$ and given by (40) for $\lambda\neq 0$.

Let us give another Markov-like theorem:

Proposition 6 *If we define*

$$\mathcal{P}_n(x) = P_{n-1}(0)P_n(x) - P_n(0)P_{n-1}(x), \quad \mathcal{Q}_n(x) = P_{n-1}(0)Q_n(x) - P_n(0)Q_{n-1}(x), \tag{59}$$

then, for an indeterminate Stieltjes moment problem, we have

$$\lim_{n\to\infty}\frac{\mathcal{Q}_n(x)}{\mathcal{P}_n(x)} = \lim_{n\to\infty}\frac{\widehat{F}_n(x)}{x\widetilde{F}_n(x)} = \int\frac{d\psi_0(t)}{x-t}, \qquad x\in V=\mathbb{C}\backslash\mathbb{R} \tag{60}$$

where the convergence is uniform for x in any compact subset of V.

Proof:
This time we use two further relations given in [1, p. 14]:

$$\begin{aligned} P_{n-1}(x) &= Q_{n-1}(0)D_n(x) - P_{n-1}(0)B_n(x),\\ Q_{n-1}(x) &= Q_{n-1}(0)C_n(x) - P_{n-1}(0)A_n(x). \end{aligned} \tag{61}$$

Combining (59) and (60) one gets

$$\mathcal{Q}_n(x) = \rho_n\, C_n(x), \qquad \mathcal{P}_n(x) = \rho_n\, D_n(x), \qquad \rho_n = \frac{(-1)^{n+1}}{\mu_n\sqrt{\pi_n}}. \tag{62}$$

In the limit $n \to \infty$ we have uniform convergence on compact subsets of V to

$$\lim_{n\to\infty} \frac{\mathcal{Q}_n(x)}{\mathcal{P}_n(x)} = \frac{C(x)}{D(x)}, \qquad x \in V. \tag{63}$$

The theorem follows from relation (40). $\square$

We have given the proofs of Markov-like theorems in the modern setting due to Nevanlinna, however let us observe that in his own setting [19] Stieltjes was aware of the existence of the measures ψ_0 and ψ_α and that they could be obtained from asymptotics.

9 A quartic example

The polynomials $F_n(c,\mu;x)$ with recurrence coefficients

$$\begin{aligned} \lambda_n &= (4n+4c+1)(4n+4c+2)^2(4n+4c+3), \\ \mu_n &= (4n+4c-1)(4n+4c)^2(4n+4c+1) + \mu\delta_{n0}, \end{aligned} \qquad c > 0, \quad \mu \in \mathbb{R}, \tag{64}$$

correspond to an indet S (hence indet H) moment problem. Their Nevanlinna matrix was given for $c = \mu = 0$ in [4] and used to obtain the border measures ψ_0 and ψ_α in closed. In the general case the Nevanlinna matrix was given in [22] but explicit measures are quite hard to get. We will show how one can recover the results for $c = \mu = 0$ using the previous Markov-like theorems.

We first need some background material. Let us define the entire functions $\delta_l(x)$, sometimes called trigonometric functions of order 4

$$\delta_l(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+l}}{(4n+l)!}, \qquad l = 0,1,2,3. \tag{65}$$

Their derivatives are

$$\delta_0' = -\delta_3, \quad \delta_1' = \delta_0, \quad \delta_2' = \delta_1, \quad \delta_3' = \delta_2 \quad \Rightarrow \quad \delta^{(4)} + \delta_l = 0, \quad l = 0,1,2,3. \tag{66}$$

the last relation explains their name. We have two simple cases

$$\delta_0(x) = \cos\left(\frac{x}{\sqrt{2}}\right)\cosh\left(\frac{x}{\sqrt{2}}\right), \qquad \delta_2(x) = \sin\left(\frac{x}{\sqrt{2}}\right)\sinh\left(\frac{x}{\sqrt{2}}\right). \tag{67}$$

We will need also the conformal mapping

$$\theta(w) = \int_0^w \frac{du}{\sqrt{1-u^4}}, \qquad \theta(1) = K_0, \tag{68}$$

which maps $\mathbb{C}\backslash \cup_{k=0}^3\ i^k[1,\infty[$ onto the square with corners $\pm\dfrac{K_0}{\sqrt{2}} \pm i\dfrac{K_0}{\sqrt{2}}$. The inversion of the mapping $\theta(w)$ involves lemniscate elliptic functions, i. e. with parameter $k^2 = 1/2$ see [26, p. 524] according to

$$w(\theta) = \frac{1}{\sqrt{2}} \frac{\text{sn } (\sqrt{2}\theta)}{\text{dn } (\sqrt{2}\theta}. \tag{69}$$

The basic tool will be the generating function

$$\sum_{n\geq 0}\frac{(c+1)_n}{(c+1/2)_n}\frac{w^{4n+4c+1}}{(4c+1)!}F_n(c,\mu;x)=\mathcal{F}(c,\mu;x;w), \tag{70}$$

with

$$\begin{aligned}\mathcal{F}(c,\mu;x;w)=&\int_0^w\frac{\delta_1(\rho(\theta(w)-\theta(u)))}{\rho}\frac{u^{4c-1}}{(4c-1)!}\,d\theta(u)\\&+\mu_0\int_0^w\frac{\delta_3(\rho(\theta(w)-\theta(u)))}{\rho^3}\frac{u^{4c+1}}{(4c+1)!}\,d\theta(u),\end{aligned}\qquad \rho=x^{1/4}. \tag{71}$$

Asymptotic analysis gives

$$F_n(c,\mu;x)\sim\frac{(4c+1)!}{4n+4c+1}\frac{(1/2)_n(c+1/2)_n}{n!\,(c+1)_n}\,\mathcal{G}(c,\mu;x), \tag{72}$$

with

$$\begin{aligned}\mathcal{G}(c,\mu;x)=&\int_0^1\delta_0(\rho(\theta(1)-\theta(u)))\frac{u^{4c-1}}{(4c-1)!}\,d\theta(u)\\&+\mu_0\int_0^1\frac{\delta_2(\rho(\theta(1)-\theta(u)))}{\rho^2}\frac{u^{4c+1}}{(4c+1)!}\,d\theta(u).\end{aligned} \tag{73}$$

So, denoting by $F_n(x)$ the polynomials corresponding to the case $c=\mu=0$ we get, by a limiting process

$$F_n(x)\sim\pi_n(c=0)\;\delta_0\left(x^{1/4}K_0/\sqrt{2}\right). \tag{74}$$

The asymptotics of $F_n^{(1)}(x)=F_n(c=1,\mu=0;x)$ is also easily obtained

$$\frac{F_{n-1}^{(1)}(x)}{\mu_1}\sim\pi_n(c=0)\int_0^1\frac{\delta_2(x^{1/4}(\theta(1)-\theta(u)))}{x^{1/2}}\,u\,d\theta(u). \tag{75}$$

Going first to the variable θ and then to $\sqrt{2}(\theta(1)-\theta)$ we are left with

$$\frac{F_{n-1}^{(1)}(x)}{\mu_1}\sim-\pi_n(c=0)\int_0^{K_0}\frac{\delta_2(x^{1/4}u/\sqrt{2}))}{x^{1/2}}\,\mathrm{cn}\,u\,\frac{du}{\sqrt{2}}. \tag{76}$$

So we can state, for the Friedrichs extension of the Jacobi matrix:

Proposition 7 *The Stieltjes transform of the measure for $F_n(x)\equiv F_n(c=0,\mu=0;x)$ reads*

$$\int\frac{d\psi_\alpha(t)}{x-t}=\frac{1}{\delta_0(x^{1/4}u/\sqrt{2})}\int_0^{K_0}\frac{\delta_2(x^{1/4}u/\sqrt{2}))}{x^{1/2}}\,\mathrm{cn}\,u\,\frac{du}{\sqrt{2}}, \tag{77}$$

and the measure

$$\psi_\alpha=\frac{4\pi}{K_0^2}\sum_{n=0}^{\infty}\frac{(2n+1)\pi}{\sinh((2n+1)\pi)}\,\epsilon_{x_n},\qquad x_n=\left(\frac{(2n+1)\pi}{K_0}\right)^4. \tag{78}$$

Proof:
The Stieltjes transform follows from (74), (76) and the first Markov-like theorem. The jumps occur at

$$x_n = \left(\frac{(2n+1)\pi}{K_0}\right)^4, \qquad n \in \mathbb{Z}. \tag{79}$$

To compute the masses one has to use the relation proved in [21, appendix]

$$\int_0^{K_0} \delta_2(x_n^{1/4}u/\sqrt{2})\,\mathrm{cn}\,u\,du = \frac{1}{4}\int_{-K_0}^{+K_0} \cos\left(x_n^{1/4}u/2\right)\frac{\mathrm{cn}\,u}{\mathrm{dn}\,u}\,du, \tag{80}$$

and this last integral is easily computed from the Fourier series of the elliptic functions. It restricts n to be positive, and gives

$$\psi_n = \frac{4\pi}{K_0^2}\frac{(2n+1)\pi}{\sinh((2n+1)\pi)} \qquad n \geq 0, \tag{81}$$

which ends the proof. $\square$.

Let us consider now the dual polynomials $\widetilde{F}_n(x) = F_n(c = 1/2, \mu = 12; x)$. Relation (72) gives

$$\widetilde{F}_n(x) \sim 3\pi_n(c=0)\frac{\delta_2(x^{1/4}K_0/\sqrt{2})}{x^{1/2}}. \tag{82}$$

Similarly we have $\widehat{F}_n = F_n(c = 1/2, \mu = 0; x)$ with the asymptotics

$$\widehat{F}_n(x) \sim -3\pi_n(c=0)\int_0^{K_0} \delta_0(x^{1/4}u/\sqrt{2})\,\mathrm{cn}\,u\,\frac{du}{\sqrt{2}}. \tag{83}$$

So we can state, for Krein's extension of the Jacobi matrix:

Proposition 8 *The Stieltjes transform of the measure for* $F_n(x) \equiv F_n(c = 0, \mu = 0; x)$ *reads*

$$\int \frac{d\psi_0(t)}{x-t} = \frac{1}{x^{1/2}\delta_2(x^{1/4}u/\sqrt{2})}\int_0^{K_0} \frac{\delta_0(x^{1/4}u/\sqrt{2}))}{x^{1/2}}\,\mathrm{cn}\,u\,\frac{du}{\sqrt{2}}, \tag{84}$$

and the measure

$$\psi_0 = \frac{\pi}{K_0^2}\,\epsilon_{x_0} + \frac{4\pi}{K_0^2}\sum_{n=1}^{\infty}\frac{2n\pi}{\sinh(2n\pi)}\,\epsilon_{x_n}, \qquad x_n = \left(\frac{2n\pi}{K_0}\right)^4. \tag{85}$$

Proof:
The Stieltjes transform follows from (82), (83) and the second Markov-like theorem. The jumps occur at

$$x_n = \left(\frac{2n\pi}{K_0}\right)^4, \qquad n \in \mathbb{Z}. \tag{86}$$

To compute the masses one has to use the relation proved in [21, appendix]

$$\int_0^{K_0} \delta_0(x_n^{1/4}u/\sqrt{u})\,\mathrm{cn}\,u\,du = \frac{1}{4}\int_{-K_0}^{+K_0} \cos\left(x_n^{1/4}u/2\right)\frac{1}{\mathrm{dn}\,u}\,du, \tag{87}$$

and this last integral is easily computed from the Fourier series of the elliptic functions. It restricts n to be positive, and gives

$$\psi_n = \frac{4\pi}{K_0^2}\frac{2n\pi}{\sinh(2n\pi)}, \qquad n \geq 0, \tag{88}$$

which ends the proof. $\square$.

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